

CROSSCORRELATION OF RUDIN-SHAPIRO-LIKE POLYNOMIALS

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ABSTRACT. We consider the Rudin-Shapiro-like polynomials, whose L^4 norms on the complex unit circle were studied by Borwein and Mossinghoff. The polynomial $f(z) = f_0 + f_1 z + \cdots + f_d z^d$ is identified with the sequence (f_0, f_1, \dots, f_d) of its coefficients. From the L^4 norm of a polynomial, one can easily calculate the autocorrelation merit factor of its associated sequence, and conversely. In this paper, we study the crosscorrelation properties of pairs of sequences associated to Rudin-Shapiro-like polynomials. We find an explicit formula for the crosscorrelation merit factor. A computer search is then used to find pairs of Rudin-Shapiro-like polynomials whose autocorrelation and crosscorrelation merit factors are simultaneously high. Pursley and Sarwate proved a bound that limits how good this combined autocorrelation and crosscorrelation performance can be. We find infinite families of polynomials whose performance approaches quite close to this fundamental limit.

1. INTRODUCTION

This paper concerns the discovery of Rudin-Shapiro-like polynomials that have exceptionally good correlation properties. Shapiro [17] recursively constructed a family of polynomials with coefficients in $\{-1, 1\}$ that are flat on the complex unit circle: the ratio of their L^∞ to L^2 norm never exceeds $\sqrt{2}$. Shapiro's polynomials were subsequently rediscovered by Rudin [13]. Littlewood used the L^4 norm on the complex unit circle in his investigation [10] into the flatness of polynomials with coefficients in $\{-1, 1\}$, which are now known as *Littlewood polynomials*. He calculated the ratio of L^4 to L^2 norm of the Rudin-Shapiro polynomials in [11, Problem 19].

Golay [3] independently developed the *merit factor*, a normalized average of the mean squared magnitude of the aperiodic autocorrelation of sequences used in remote sensing and communications networks. Eventually it was discovered that determining the L^4 norm on the unit circle of a polynomial is tantamount to determining Golay's merit factor for the sequence of coefficients of that polynomial (see [6, eq. (4.1)]). We shall soon make precise this connection between the analytic behavior of polynomials on the complex unit circle and the correlation behavior of their associated sequences.

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Inspired by the work of Littlewood on the Rudin-Shapiro polynomials, Borwein and Mossinghoff [2] recursively define sequences $f_0(z), f_1(z), \dots$ of polynomials, where $f_0(z)$ is any Littlewood polynomial and the rest of the polynomials are obtained via a recursion of the form

$$(1) \quad f_{n+1}(z) = f_n(z) + \sigma_n z^{1+\deg f_n} f_n^\dagger(-z),$$

where $\sigma_n \in \{-1, 1\}$ represents an arbitrary sign that can differ at each stage of the recursion, and where for any polynomial $a(z) = a_0 + a_1 z + \dots + a_d z^d \in \mathbb{C}[z]$ of degree d , the polynomial $a^\dagger(z)$ denotes the *conjugate reciprocal polynomial* of $a(z)$, which is $\overline{a_d} + \overline{a_{d-1}}z + \dots + \overline{a_0}z^d$.¹

In this paper, we relax the condition that the initial polynomial $f_0(z)$ be a Littlewood polynomial. We allow the $f_0(z)$ to be a polynomial in $\mathbb{C}[z]$, and only impose the condition that $f_0(z)$ has a nonzero constant coefficient. This ensures that $f_0^\dagger(z)$ has the same degree as $f_0(z)$, so that when we construct $f_0(z), f_1(z), \dots$ via recursion (1), a straightforward induction shows that every f_n has a nonzero constant coefficient and

$$(2) \quad 1 + \deg f_n = 2^n(1 + \deg f_0)$$

for every n . We call the sequence $\sigma = \sigma_0, \sigma_1, \dots$ of numbers in $\{-1, 1\}$ that occur in our recursion the *sign sequence* for that recursion. We call $f_0(z)$ the *seed*, and the sequence $f_0(z), f_1(z), \dots$ of polynomials obtained from the seed by applying the recursion is called the *stem* associated to seed $f_0(z)$ and sign sequence σ . Any stem obtained from a seed $f_0(z) \in \mathbb{C}[z]$ with nonzero constant coefficient is also called a *sequence of Rudin-Shapiro-like polynomials*.

Borwein and Mossinghoff [2] study the L^4 norm of Rudin-Shapiro-like Littlewood polynomials on the complex unit circle, or equivalently, the autocorrelation merit factor of these polynomials. We now describe the correspondence between polynomials and sequences, and the relation of L^p norms on the complex unit circle to correlation.

In this paper, by a *sequence of length ℓ* we mean some $(a_0, a_1, \dots, a_{\ell-1}) \in \mathbb{C}^\ell$. Most researchers are especially interested in sequences that are *unimodular*, that is, whose terms are all of unit magnitude, and are most of all interested in sequences that are *binary*, that is, whose terms lie in $\{-1, 1\}$. We always identify the polynomial $a(z) = a_0 + a_1 z + \dots + a_{\ell-1} z^{\ell-1} \in \mathbb{C}[z]$ of degree $\ell - 1$ with the sequence $a = (a_0, a_1, \dots, a_{\ell-1})$ of length ℓ . With this identification, binary sequences correspond to Littlewood polynomials. Usually it is easier to work with sequence length rather than polynomial degree, and using our identification of sequences with polynomials, we define the *length* of a polynomial to be the length of the sequence associated with the polynomial, that is, $\text{len } a = 1 + \deg a$. Notice that when we have a

¹In fact, Borwein-Mossinghoff use the *reciprocal polynomial* $a^*(z) = a_d + a_{d-1}z + \dots + a_0$, but since they are working with polynomials with real coefficients, this is the same as $a^\dagger(z)$. In this paper, it was found that using $a^\dagger(z)$ instead of $a^*(z)$ gives the natural generalization of their recursion for polynomials with non-real coefficients.

stem $f_0(z), f_1(z), \dots$ generated from a seed $f_0(z) \in \mathbb{C}[z]$ with nonzero constant coefficient via recursion (1), using length rather than degree simplifies relation (2) to

$$(3) \quad \text{len } f_n = 2^n \text{len } f_0.$$

If $f = (f_0, f_1, \dots, f_{\ell-1})$ and $g = (g_0, g_1, \dots, g_{\ell-1})$ are two sequences of length ℓ and $s \in \mathbb{Z}$, then we define the *aperiodic crosscorrelation of f and g at shift s* to be

$$C_{f,g}(s) = \sum_{j \in \mathbb{Z}} f_j \overline{g_{j+s}},$$

where we take $f_j = g_j = 0$ whenever $j \notin \{0, 1, \dots, \ell-1\}$.

Autocorrelation is crosscorrelation of a sequence with itself, so the *aperiodic autocorrelation of f at shift s* is just $C_{f,f}(s)$. If the terms of f are complex numbers of unit magnitude, then $C_{f,f}(0) = \text{len}(f)$.

Autocorrelation and crosscorrelation are studied extensively because of their importance in communications networks: see [15, 14, 4, 8, 7, 5, 16] for some overviews. It is desirable to have sequences whose autocorrelation values at all nonzero shifts are small in magnitude, and it is desirable to have pairs of sequences whose crosscorrelation values at all shifts are small in magnitude.

For a polynomial $a(z) = a_0 + a_1 z + \dots + a_\ell z^\ell \in \mathbb{C}[z]$, we define $\overline{a(z)}$ to be the Laurent polynomial $\overline{a_0} + \overline{a_1} z^{-1} + \dots + \overline{a_{\ell-1}} z^{-(\ell-1)}$. (We identify \bar{z} with z^{-1} because we are concerned with the properties of our polynomials on the complex unit circle.) If $f(z)$ and $g(z)$ are polynomials in $\mathbb{C}[z]$, then it is not hard to show that the values of the crosscorrelation between their associated sequences at all shifts are recorded in the following product of Laurent polynomials:

$$f(z) \overline{g(z)} = \sum_{s \in \mathbb{Z}} C_{f,g}(-s) z^s.$$

The *crosscorrelation demerit factor* of f and g is defined to be

$$\text{CDF}(f, g) = \frac{\sum_{s \in \mathbb{Z}} |C_{f,g}(s)|^2}{|C_{f,f}(0)| \cdot |C_{g,g}(0)|}.$$

Its reciprocal, the *crosscorrelation merit factor*, is defined as $\text{CMF}(f, g) = 1/\text{CDF}(f, g)$. A low demerit factor (or equivalently, high merit factor) indicates a sequence pair whose crosscorrelation values are collectively low, hence desirable. The *autocorrelation demerit factor of f* is much like the crosscorrelation demerit factor, but omits $|C_{f,f}(0)|^2$ in the numerator:

$$(4) \quad \text{ADF}(f) = \frac{\sum_{s \in \mathbb{Z}, s \neq 0} |C_{f,f}(s)|^2}{|C_{f,f}(0)|^2} = \text{CDF}(f, f) - 1,$$

and the *autocorrelation merit factor* is its reciprocal $\text{AMF}(f) = 1/\text{ADF}(f)$. The autocorrelation merit factor defined here is Golay's original merit factor, introduced in [3].

If $f(z) \in \mathbb{C}[z, z^{-1}]$ is a Laurent polynomial, and p is a real number with $p \geq 1$, then we define the L^p norm of $f(z)$ on the complex unit circle to be

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}.$$

One can show (see [9, Section V]) that

$$(5) \quad \text{CDF}(f, g) = \frac{\|fg\|_2^2}{\|f\|_2^2 \|g\|_2^2},$$

and

$$\text{ADF}(f) = \text{CDF}(f, f) - 1 = \frac{\|f\|_4^4}{\|f\|_2^4} - 1.$$

Borwein and Mossinghoff [2, Theorem 1 and Corollary 1] explicitly calculate the autocorrelation demerit factors for Rudin-Shapiro-like Littlewood polynomials and determine their asymptotic behavior.

Theorem 1.1 (Borwein-Mossinghoff, 2000). *If f_0, f_1, \dots is a sequence of Rudin-Shapiro-like polynomials generated from any Littlewood polynomial $f_0(z)$ via recursion (1), then*

$$\lim_{n \rightarrow \infty} \text{ADF}(f_n) = -1 + \frac{2}{3} \cdot \frac{\|f_0\|_4^4 + \|f_0 \tilde{f}_0\|_2^2}{\|f_0\|_2^4} \geq \frac{1}{3},$$

where $\tilde{f}_0(z)$ is the polynomial $f_0(-z)$.

Borwein and Mossinghoff [2, Section 3] go on to find examples where the limiting autocorrelation demerit factor reaches the lower bound of $1/3$, so that well-chosen families of Rudin-Shapiro-like polynomials can reach asymptotic autocorrelation merit factors as high as 3.

We are interested in both autocorrelation and crosscorrelation merit factors. It turns out that there are limits to how good one can simultaneously make autocorrelation and crosscorrelation performance. Pursley and Sarwate [12, eqs. (3),(4)] proved a bound that relates autocorrelation and crosscorrelation demerit factors:

$$(6) \quad |\text{CDF}(f, g) - 1| \leq \sqrt{\text{ADF}(f) \text{ADF}(g)}.$$

We define the *Pursley-Sarwate criterion* of f and g to be

$$(7) \quad \text{PSC}(f, g) = \sqrt{\text{ADF}(f) \text{ADF}(g)} + \text{CDF}(f, g),$$

so that (6) implies that

$$(8) \quad \text{PSC}(f, g) \geq 1.$$

Since we want sequence pairs with low mutual crosscorrelation and where both sequences individually have low autocorrelation, we would like to find f and g with $\text{PSC}(f, g)$ as close to 1 as possible.

In pursuit of this goal, we found a formula for the asymptotic crosscorrelation demerit factor of a family of pairs of Rudin-Shapiro-like sequences.

This formula specializes to give information about autocorrelation that generalizes the results of Theorem 1.1 to embrace polynomials with coefficients other than -1 and 1 .

Theorem 1.2. *Let $f_0, g_0 \in \mathbb{C}[z]$ be polynomials of equal length having nonzero constant coefficients. If f_0, f_1, \dots and g_0, g_1, \dots are sequences of Rudin-Shapiro-like polynomials generated from f_0 and g_0 via recursion (1), then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{ADF}(f_n) &= -1 + \frac{2}{3} \cdot \frac{\|f_0\|_4^4 + \|f_0 \tilde{f}_0\|_2^2}{\|f_0\|_2^4} \geq \frac{1}{3}, \\ \lim_{n \rightarrow \infty} \text{ADF}(g_n) &= -1 + \frac{2}{3} \cdot \frac{\|g_0\|_4^4 + \|g_0 \tilde{g}_0\|_2^2}{\|g_0\|_2^4} \geq \frac{1}{3}, \\ \lim_{n \rightarrow \infty} \text{CDF}(f_n, g_n) &= \frac{2\|f_0 g_0\|_2^2 + \|f_0 \tilde{g}_0\|_2^2 + \text{Re} \int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0}}{3\|f_0\|_2^2 \|g_0\|_2^2}, \end{aligned}$$

where $\tilde{f}_0(z)$ and $\tilde{g}_0(z)$ are respectively the polynomials $f_0(-z)$ and $g_0(-z)$, and

$$\int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0} = \frac{1}{2\pi} \int_0^{2\pi} f_0(e^{i\theta}) \tilde{f}_0(e^{i\theta}) \overline{g_0(e^{i\theta}) \tilde{g}_0(e^{i\theta})} d\theta.$$

This theorem is proved in Corollary 2.7 in Section 2. We then use the formula in Theorem 1.2 and computational searches (see Section 4) to find families of pairs Rudin-Shapiro-like Littlewood polynomials whose asymptotic Pursley-Sarwate criterion is as low as $331/300 = 1.10333\dots$, which is quite close to the absolute lower bound of (8). In contrast, the typical Pursley-Sarwate criterion of randomly selected long binary sequences is about 2, and high-performance sequence pairs constructed from finite field characters have been found with asymptotic Pursley-Sarwate criterion of $7/6$ (see [9, §II.E, §IV.D] and [1, eq.(6)]). As far as the authors know, the limiting value of $331/300$ found here is the lowest asymptotic value reported to date.

The rest of this paper is organized as follows. The goal of Section 2 is to prove Theorem 1.2 above. This is accomplished by finding recursive relations between various L^p norms associated with our Rudin-Shapiro-like polynomials that arise from the original recursion (1).

Section 3 examines groups of symmetries that preserve the asymptotic correlation behavior when applied to our Rudin-Shapiro-like polynomials. These are helpful in abbreviating computational searches (reported later in Section 4) for polynomials with good autocorrelation and crosscorrelation performance. We organize the good polynomials that we find into orbits modulo the action of our symmetry groups, which makes our reports shorter and more intelligible.

In Section 4, we present some examples of Rudin-Shapiro-like Littlewood polynomials f_0, f_1, \dots and g_0, g_1, \dots such that $\lim_{n \rightarrow \infty} \text{PSC}(f_n, g_n)$ is low,

which implies simultaneously good autocorrelation and crosscorrelation performance. This includes the families of polynomials with the exceptionally low asymptotic Pursley-Sarwate criterion value reported above.

2. ASYMPTOTIC CROSSCORRELATION FORMULA

In this section we prove Theorem 1.2, the main theoretical result of this paper. First we set down some notational conventions.

Throughout this paper, we let $\mathbb{C}[z, z^{-1}]$ denote the ring of Laurent polynomials with coefficients from \mathbb{C} . Because we are working with polynomials on the complex unit circle, we use $\overline{a(z)}$ as a shorthand for $\sum_{j \in \mathbb{Z}} \overline{a_j} z^{-j}$, $\operatorname{Re}(a(z))$ as a shorthand for $\frac{1}{2}(a(z) + \overline{a(z)})$, and $|a(z)|^2$ as a shorthand for $a(z)\overline{a(z)}$. We also use $\widetilde{a}(z)$ as a shorthand for $a(-z)$. Also recall from the Introduction that if $a(z) = a_0 + a_1 z + \cdots + a_{d-1} z^{d-1}$ is a polynomial of degree d in $\mathbb{C}[z]$, then $a^\dagger(z)$ denotes the conjugate reciprocal polynomial of $a(z)$, that is, $a^\dagger(z) = \overline{a_d} + \cdots + \overline{a_1} z^{d-1} + \overline{a_0}$.

We first note how the transformations $a \mapsto \widetilde{a}$, $a \mapsto \overline{a}$, and $a \mapsto a^\dagger$ relate to and interact with each other.

Lemma 2.1. *If $f(z) \in \mathbb{C}[z]$, then*

- (i). $f^\dagger(z) = z^{\deg f} \overline{f(z)}$,
- (ii). $\widetilde{(f^\dagger)}(z) = (-z)^{\deg f} \overline{\widetilde{f}(z)} = (-1)^{\deg f} \cdot (\widetilde{f})^\dagger(z)$, and
- (iii). $\overline{f^\dagger(z)} = z^{-\deg f} f(z)$.

Proof. For part (i), note that if $d = \deg f$ and $f(z) = f_0 + f_1 z + \cdots + f_d z^d$, then by the definition of the conjugate reciprocal, we have

$$\begin{aligned} f^\dagger(z) &= \overline{f_d} + \cdots + \overline{f_1} z^{d-1} + \overline{f_0} z^d \\ &= z^d (\overline{f_d} z^{-d} + \cdots + \overline{f_1} z^{-1} + \overline{f_0}) \\ &= z^d \overline{f(z)}. \end{aligned}$$

For part (ii), use part (i) to see that

$$\begin{aligned} \widetilde{(f^\dagger)}(z) &= (-z)^{\deg f} \overline{\widetilde{f}(-z)} \\ &= (-z)^{\deg f} \overline{\widetilde{f}(z)}, \end{aligned}$$

and then

$$\begin{aligned} (-z)^{\deg f} \overline{\widetilde{f}(z)} &= (-1)^{\deg f} \cdot z^{\deg f} \overline{\widetilde{f}(z)} \\ &= (-1)^{\deg f} \cdot (\widetilde{f})^\dagger(z), \end{aligned}$$

where we used part (i) again in the second equality.

For part (iii), use part (i) to see that

$$\begin{aligned} \overline{f^\dagger(z)} &= \overline{z^{\deg f} \overline{f(z)}} \\ &= z^{-\deg f} f(z). \end{aligned}$$

□

If $a(z) \in \mathbb{C}[z, z^{-1}]$, say $a(z) = \sum_{j \in \mathbb{Z}} a_j z^j$, then we use $\int a(z)$ as a shorthand for a_0 . This is because if we actually perform integration on the complex unit circle, we obtain a_0 :

$$\frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) d\theta = a_0.$$

In particular note that

$$\|a(z)\|_2^2 = \int a(z) \overline{a(z)} = \sum_{j \in \mathbb{Z}} |a_j|^2.$$

It will be important to know that replacing a with \tilde{a} changes neither integrals nor norms.

Lemma 2.2.

- (i). For any $f(z) \in \mathbb{C}[z, z^{-1}]$, we have $\int \tilde{f}(z) = \int f(z)$.
- (ii). For any $f(z) \in \mathbb{C}[z, z^{-1}]$ and any $p \in \mathbb{R}$ with $p \geq 1$, we have $\|\tilde{f}(z)\|_p = \|f(z)\|_p$.

Proof. If $g(z)$ is any function that is integrable on the complex unit circle, then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} g(-e^{i\theta}) d\theta &= \frac{1}{2\pi} \int_0^\pi g(e^{i(\theta+\pi)}) d\theta + \frac{1}{2\pi} \int_\pi^{2\pi} g(e^{i(\theta-\pi)}) d\theta \\ &= \frac{1}{2\pi} \int_\pi^{2\pi} g(e^{i\eta}) d\eta + \frac{1}{2\pi} \int_0^\pi g(e^{i\eta}) d\eta \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) d\theta. \end{aligned}$$

This proves parts (i) and (ii), where we set $g(z) = f(z)$ or $g(z) = |f(z)|^p$, respectively. \square

If we have two sequences f_0, f_1, \dots and g_0, g_1, \dots of Rudin-Shapiro-like polynomials constructed via recursion (1), then the following lemma tells us how $\|f_{n+1}g_{n+1}\|_2^2$ is related to $\|f_n g_n\|_2^2$. In view of (5), this is telling us how $\text{CDF}(f_n, g_n)$ changes in one step of the recursion.

Lemma 2.3. Suppose $f_n(z), g_n(z) \in \mathbb{C}[z]$ are polynomials of length ℓ , and let $f_{n+1}(z) = f_n(z) + \sigma_n z^\ell f_n^\dagger(-z)$ and $g_{n+1}(z) = g_n(z) + \tau_n z^\ell g_n^\dagger(-z)$, where $\sigma_n, \tau_n \in \{-1, 1\}$. Then $\|f_{n+1}\|_2^2 = 2\|f_n\|_2^2$, $\|g_{n+1}\|_2^2 = 2\|g_n\|_2^2$, and if we define

- (i). $u_j = \|f_j g_j\|_2^2$,
- (ii). $v_j = \|f_j \tilde{g}_j\|_2^2$,
- (iii). $w_j = \text{Re} \int f_j \tilde{f}_j \overline{g_j \tilde{g}_j}$

for $j \in \{n, n+1\}$, then

- (i). $u_{n+1} = 2u_n + 2v_n + 2\sigma_n \tau_n w_n$,
- (ii). $v_{n+1} = 2u_n + 2v_n - 2\sigma_n \tau_n w_n$, and
- (iii). $w_{n+1} = 2\sigma_n \tau_n u_n - 2\sigma_n \tau_n v_n + 2w_n$.

Proof. First observe that Lemma 2.1(ii) shows that

$$(9) \quad \begin{aligned} f_{n+1}(z) &= f_n(z) + \sigma_n(-1)^{\ell-1} z^{2\ell-1} \overline{\widetilde{f}_n(z)} \\ g_{n+1}(z) &= g_n(z) + \tau_n(-1)^{\ell-1} z^{2\ell-1} \overline{\widetilde{g}_n(z)}, \end{aligned}$$

and so

$$\begin{aligned} \|f_{n+1}\|_2^2 &= \int \left| f_n + \sigma_n(-1)^{\ell-1} z^{2\ell-1} \overline{\widetilde{f}_n} \right|^2 \\ &= \int |f_n|^2 + |\widetilde{f}_n|^2 + 2 \operatorname{Re}(\sigma_n(-1)^{\ell-1} z^{2\ell-1} \overline{f_n \widetilde{f}_n}) \\ &= 2\|f_n\|_2^2 + 2\sigma_n(-1)^{\ell-1} \operatorname{Re} \int z^{2\ell-1} \overline{f_n \widetilde{f}_n} \end{aligned}$$

where the last equality uses Lemma 2.2(ii). Now observe that $z^{2\ell-1} \overline{f_n \widetilde{f}_n}$ is a Laurent polynomial whose terms all have positive powers of z (because f_n is a polynomial of degree $\ell-1$), and so the last integral is zero. Thus we obtain the desired result that $\|f_{n+1}\|_2^2 = 2\|f_n\|_2^2$. If one replaces every instance of f with g in the above, one obtains a proof that $\|g_{n+1}\|_2^2 = 2\|g_n\|_2^2$.

Now we prove the recursions involving u_n , v_n , and w_n . First of all,

$$\begin{aligned} u_{n+1} &= \|f_{n+1}g_{n+1}\|_2^2 \\ &= \frac{1}{2}\|f_{n+1}g_{n+1}\|_2^2 + \frac{1}{2}\|\widetilde{f}_{n+1}\widetilde{g}_{n+1}\|_2^2 \\ &= \frac{1}{2}\left\| \left(f_n + \sigma_n(-1)^{\ell-1} z^{2\ell-1} \overline{\widetilde{f}_n} \right) \left(g_n + \tau_n(-1)^{\ell-1} z^{2\ell-1} \overline{\widetilde{g}_n} \right) \right\|_2^2 \\ &\quad + \frac{1}{2}\left\| \left(\widetilde{f}_n - \sigma_n(-1)^{\ell-1} z^{2\ell-1} \overline{f_n} \right) \left(\widetilde{g}_n - \tau_n(-1)^{\ell-1} z^{2\ell-1} \overline{g_n} \right) \right\|_2^2 \\ &= 2\|f_n g_n\|_2^2 + 2\|f_n \widetilde{g}_n\|_2^2 + 2\sigma_n \tau_n \operatorname{Re} \int f_n \widetilde{f}_n \overline{g_n \widetilde{g}_n} \\ &\quad + 2\sigma_n \tau_n \operatorname{Re} \int z^{2-4\ell} \widetilde{f}_n \widetilde{f}_n g_n \widetilde{g}_n \\ &= 2u_n + 2v_n + 2\sigma_n \tau_n w_n + 2\sigma_n \tau_n \operatorname{Re} \int z^{2-4\ell} \widetilde{f}_n \widetilde{f}_n g_n \widetilde{g}_n, \end{aligned}$$

where the first equality is the definition of u_{n+1} , the second equality uses Lemma 2.2(ii), the third equality uses (9), and the fourth equality uses technical Lemma 2.10, which appears at the end of this section. Then note that $z^{2-4\ell} \widetilde{f}_n \widetilde{f}_n g_n \widetilde{g}_n$ is a Laurent polynomial whose terms all have negative powers of z (because f_n and g_n are polynomials of degree $\ell-1$), so that the last integral in our chain of equalities is zero, giving us the desired result.

We also have

$$\begin{aligned}
v_{n+1} &= \|f_{n+1}\tilde{g}_{n+1}\|_2^2 \\
&= \frac{1}{2}\|f_{n+1}\tilde{g}_{n+1}\|_2^2 + \frac{1}{2}\|\tilde{f}_{n+1}g_{n+1}\|_2^2 \\
&= \frac{1}{2}\left\|\left(f_n + \sigma_n(-1)^{\ell-1}z^{2\ell-1}\tilde{f}_n\right)\left(\tilde{g}_n - \tau_n(-1)^{\ell-1}z^{2\ell-1}\overline{g_n}\right)\right\|_2^2 \\
&\quad + \frac{1}{2}\left\|\left(\tilde{f}_n - \sigma_n(-1)^{\ell-1}z^{2\ell-1}\overline{f_n}\right)\left(g_n + \tau_n(-1)^{\ell-1}z^{2\ell-1}\tilde{g}_n\right)\right\|_2^2 \\
&= 2\|f_n\tilde{g}_n\|_2^2 + 2\|f_ng_n\|_2^2 + 2\sigma_n(-\tau_n)\operatorname{Re}\int f_n\tilde{f}_n\overline{g_ng_n} \\
&\quad + 2\sigma_n(-\tau_n)\operatorname{Re}\int z^{2-4\ell}f_n\tilde{f}_n\tilde{g}_ng_n \\
&= 2u_n + 2v_n - 2\sigma_n\tau_nw_n - 2\sigma_n\tau_n\operatorname{Re}\int z^{2-4\ell}f_n\tilde{f}_ng_n\tilde{g}_n,
\end{aligned}$$

where the first equality is the definition of v_{n+1} , the second equality uses Lemma 2.2(ii), the third equality uses (9), and the fourth equality uses technical Lemma 2.10, which appears at the end of this section. Then note that $z^{2-4\ell}f_n\tilde{f}_ng_n\tilde{g}_n$ is a Laurent polynomial whose terms all have negative powers of z (because f_n and g_n are polynomials of degree $\ell-1$), so that the last integral in our chain of equalities is zero, giving us the desired result.

Finally, we use similar arguments to obtain

$$\begin{aligned}
w_{n+1} &= \operatorname{Re}\int f_{n+1}\tilde{f}_{n+1}\overline{g_{n+1}\tilde{g}_{n+1}} \\
&= \operatorname{Re}\int \left(f_n + \sigma_n(-1)^{\ell-1}z^{2\ell-1}\tilde{f}_n\right)\left(\tilde{f}_n - \sigma_n(-1)^{\ell-1}z^{2\ell-1}\overline{f_n}\right) \\
&\quad \times \left(\overline{g_n} + \tau_n(-1)^{\ell-1}z^{1-2\ell}\tilde{g}_n\right)\left(\tilde{g}_n - \tau_n(-1)^{\ell-1}z^{1-2\ell}g_n\right) \\
&= \operatorname{Re}\int \left[f_n\tilde{f}_n - z^{4\ell-2}\overline{f_n\tilde{f}_n} + \sigma_n(-1)^{\ell-1}z^{2\ell-1}\left(|\tilde{f}_n|^2 - |f_n|^2\right)\right] \\
&\quad \times \left[\overline{g_n\tilde{g}_n} - z^{2-4\ell}g_n\tilde{g}_n + \tau_n(-1)^{\ell-1}z^{1-2\ell}\left(|\tilde{g}_n|^2 - |g_n|^2\right)\right] \\
&= I_1 + \sigma_n\tau_nI_2 + \tau_n(-1)^{\ell-1}I_3 + \sigma_n(-1)^{\ell-1}I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \operatorname{Re}\int \left(f_n\tilde{f}_n - z^{4\ell-2}\overline{f_n\tilde{f}_n}\right)\left(\overline{g_n\tilde{g}_n} - z^{2-4\ell}g_n\tilde{g}_n\right) \\
I_2 &= \operatorname{Re}\int \left(|\tilde{f}_n|^2 - |f_n|^2\right)\left(|\tilde{g}_n|^2 - |g_n|^2\right) \\
I_3 &= \operatorname{Re}\int \left(f_n\tilde{f}_n - z^{4\ell-2}\overline{f_n\tilde{f}_n}\right)z^{1-2\ell}\left(|\tilde{g}_n|^2 - |g_n|^2\right) \\
I_4 &= \operatorname{Re}\int z^{2\ell-1}\left(|\tilde{f}_n|^2 - |f_n|^2\right)\left(\overline{g_n\tilde{g}_n} - z^{2-4\ell}g_n\tilde{g}_n\right).
\end{aligned}$$

Now we compute each of these four integrals.

$$\begin{aligned} I_1 &= \operatorname{Re} \int \left(f_n \tilde{f}_n - z^{4\ell-2} \overline{f_n \tilde{f}_n} \right) \left(\overline{g_n \tilde{g}_n} - z^{2-4\ell} g_n \tilde{g}_n \right) \\ &= 2 \operatorname{Re} \int f_n \tilde{f}_n \overline{g_n \tilde{g}_n} - 2 \operatorname{Re} \int z^{2-4\ell} f_n \tilde{f}_n g_n \tilde{g}_n, \end{aligned}$$

and note that $z^{2-4\ell} f_n \tilde{f}_n g_n \tilde{g}_n$ is a Laurent polynomial whose terms all have negative powers of z (because f_n and g_n are of length $\ell-1$), so that the last integral is zero, and thus

$$I_1 = 2w_n.$$

Then

$$\begin{aligned} I_2 &= \|\tilde{f}_n \tilde{g}_n\|_2^2 + \|f_n g_n\|_2^2 - \|\tilde{f}_n g_n\|_2^2 - \|f_n \tilde{g}_n\|_2^2 \\ &= 2u_n - 2v_n, \end{aligned}$$

where the second equality is due to Lemma 2.2(ii).

Let us examine the integrand in the definition of I_3 , which is

$$h = \left(z^{1-2\ell} f_n \tilde{f}_n - z^{2\ell-1} \overline{f_n \tilde{f}_n} \right) (|\tilde{g}_n|^2 - |g_n|^2).$$

If one conjugates this, one obtains

$$\left(z^{2\ell-1} \overline{f_n \tilde{f}_n} - z^{1-2\ell} f_n \tilde{f}_n \right) (|\tilde{g}_n|^2 - |g_n|^2),$$

which is just $-h$. So h has purely imaginary values on the unit circle, and thus $I_3 = \operatorname{Re} \int h = 0$. The same argument shows that $I_4 = 0$, and so, putting all our results together, we have

$$\begin{aligned} w_{n+1} &= I_1 + \sigma_n \tau_n I_2 + \tau_n (-1)^{\ell-1} I_3 + \sigma_n (-1)^{\ell-1} I_4 \\ &= 2w_n + \sigma_n \tau_n (2u_n - 2v_n) + 0 + 0. \end{aligned}$$

□

The above lemma allows us to compute crosscorrelation demerit factors for pairs of Rudin-Shapiro-like polynomials constructed via recursion (1).

Theorem 2.4. *Let $f_0, g_0 \in \mathbb{C}[z]$ be polynomials of equal length having nonzero constant coefficients. Let $\sigma_0, \sigma_1, \dots$ be a sequence of values from $\{-1, 1\}$, and suppose that $f_n(z)$ and $g_n(z)$ are defined recursively for all $n \in \mathbb{N}$ by*

$$\begin{aligned} f_{n+1}(z) &= f_n(z) + \sigma_n z^{\operatorname{len} f_n} f_n^\dagger(-z) \\ g_{n+1}(z) &= g_n(z) + \sigma_n z^{\operatorname{len} g_n} g_n^\dagger(-z). \end{aligned}$$

Then

$$\begin{aligned} \frac{\|f_n g_n\|_2^2}{\|f_n\|_2^2 \|g_n\|_2^2} &= \frac{2\|f_0 g_0\|_2^2 + \|f_0 \tilde{g}_0\|_2^2 + \operatorname{Re} \int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0}}{3\|f_0\|_2^2 \|g_0\|_2^2} \\ &\quad + \left(-\frac{1}{2}\right)^n \frac{\|f_0 g_0\|_2^2 - \|f_0 \tilde{g}_0\|_2^2 - \operatorname{Re} \int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0}}{3\|f_0\|_2^2 \|g_0\|_2^2}. \end{aligned}$$

Proof. Since f_0 and g_0 have nonzero constant coefficients and are of the same length, induction shows that for every n , the polynomials f_n and g_n have nonzero constant coefficients and both are of length $2^n \text{len } f_0 = 2^n \text{len } g_0$, as observed in (3) in the Introduction. Thus we may apply Lemma 2.3 repeatedly to the pairs (f_n, g_n) for every n .

Let u_j , v_j , and w_j be as defined in Lemma 2.3. We want to calculate u_n , and the lemma says that

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = A \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}.$$

Now $A = B\Lambda B^{-1}$, where

$$B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

So

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} &= A^n \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \\ &= B\Lambda^n B^{-1} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}, \end{aligned}$$

and so

$$\|f_n g_n\|_2^2 = u_n = \frac{4^n(2u_0 + v_0 + w_0) + (-2)^n(u_0 - v_0 - w_0)}{3}.$$

Repeated use of Lemma 2.3 also shows that $\|f_n\|_2^2 = 2^n \|f_0\|_2^2$ and $\|g_n\|_2^2 = 2^n \|g_0\|_2^2$, and these norms are nonzero since f_0 and g_0 are nonzero, so that

$$\frac{\|f_n g_n\|_2^2}{\|f_n\|_2^2 \|g_n\|_2^2} = \frac{(2u_0 + v_0 + w_0) + (-1/2)^n(u_0 - v_0 - w_0)}{3\|f_0\|_2^2 \|g_0\|_2^2},$$

and when one substitutes the values of u_0 , v_0 , and w_0 as defined in Lemma 2.3, then one obtains the desired result. \square

If $f_0 = g_0$, we are considering autocorrelation. When we specialize to this case and also specialize to the case where f_0 is a Littlewood polynomial, then we recover the results of Borwein and Mossinghoff [2, Theorem 1 and Corollary 1].

Corollary 2.5 (Borwein-Mossinghoff (2000)). *Suppose that $f_0(z)$ is a Littlewood polynomial and that $f_n(z)$ is defined recursively for all $n \in \mathbb{N}$ by*

$$f_{n+1}(z) = f_n(z) + z^{\text{len } f_n} f_n^\dagger(-z).$$

Then

$$\frac{\|f_n\|_4^4}{\|f_n\|_2^4} = \frac{2}{3} \cdot \frac{\|f_0\|_4^4 + \|f_0 \tilde{f}_0\|_2^2}{\|f_0\|_2^4} + \left(-\frac{1}{2}\right)^n \cdot \frac{1}{3} \cdot \frac{\|f_0\|_4^4 - 2\|f_0 \tilde{f}_0\|_2^2}{\|f_0\|_2^4}.$$

Remark 2.6. Borwein and Mossinghoff use $\|f_0 \tilde{f}_0^*\|_2^2$ instead of our $\|f_0 \tilde{f}_0\|_2^2$, but it is not hard to see that these are equal because $f_0^*(-z) = \tilde{f}^\dagger(z) = (-z)^{\deg f} \overline{\tilde{f}(z)}$ for any Littlewood polynomial (see Lemma 2.1(ii)).

Our results now allow us to compute limiting autocorrelation and cross-correlation demerit factors. The following corollary contains all the results that we presented in Theorem 1.2 in the Introduction.

Corollary 2.7. *Let $f_0, g_0 \in \mathbb{C}[z]$ be polynomials of equal length having nonzero constant coefficients. Let $\sigma_0, \sigma_1, \dots$ be a sequence of values from $\{-1, 1\}$, and suppose that $f_n(z)$ and $g_n(z)$ are defined recursively for all $n \in \mathbb{N}$ by*

$$\begin{aligned} f_{n+1}(z) &= f_n(z) + \sigma_n z^{\text{len } f_n} f_n^\dagger(-z) \\ g_{n+1}(z) &= g_n(z) + \sigma_n z^{\text{len } g_n} g_n^\dagger(-z). \end{aligned}$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{ADF}(f_n) &= -1 + \frac{2}{3} \cdot \frac{\|f_0\|_4^4 + \|f_0 \tilde{f}_0\|_2^2}{\|f_0\|_2^4} \geq \frac{1}{3}, \\ \lim_{n \rightarrow \infty} \text{ADF}(g_n) &= -1 + \frac{2}{3} \cdot \frac{\|g_0\|_4^4 + \|g_0 \tilde{g}_0\|_2^2}{\|g_0\|_2^4} \geq \frac{1}{3}, \\ \lim_{n \rightarrow \infty} \text{CDF}(f_n, g_n) &= \frac{2\|f_0 g_0\|_2^2 + \|f_0 \tilde{g}_0\|_2^2 + \text{Re} \int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0}}{3\|f_0\|_2^2 \|g_0\|_2^2}, \end{aligned}$$

so that

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{PSC}(f_n, g_n) &= \frac{2\|f_0 g_0\|_2^2 + \|f_0 \tilde{g}_0\|_2^2 + \text{Re} \int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0}}{3\|f_0\|_2^2 \|g_0\|_2^2} \\ &\quad + \frac{\sqrt{\left(2\|f_0\|_4^4 + 2\|f_0 \tilde{f}_0\|_2^2 - 3\|f_0\|_2^4\right) \left(2\|g_0\|_4^4 + 2\|g_0 \tilde{g}_0\|_2^2 - 3\|g_0\|_2^4\right)}}{3\|f_0\|_2^2 \|g_0\|_2^2}. \end{aligned}$$

Proof. The limiting crosscorrelation demerit factor is clear from Theorem 2.4 since the ratio norms calculated there is the crosscorrelation demerit factor by (5). For the limiting autocorrelation demerit factors, one again uses Theorem 2.4, but now one sets $f_n = g_n$ for all n in that theorem, and combines the result thus obtained with the fact from (4) that $\text{ADF}(f_n) = \text{CDF}(f_n, f_n) - 1$ along with the observations that $\|f_0 f_0\|_2^2 = \|f_0\|_4^4$ and

$\operatorname{Re} \int f_0 \tilde{f}_0 \overline{f_0 \tilde{f}_0} = \|f_0 \tilde{f}_0\|_2^2$. The limiting Pursley-Sarwate criterion follows immediately from the definition in (7) and the limits on the autocorrelation and crosscorrelation demerit factors.

To obtain the lower bounds on the limiting autocorrelation demerit factors, one notes that for any $f(z) \in \mathbb{C}[z]$, we have

$$\begin{aligned} \|f\|_4^4 + \|f\tilde{f}\|_2^2 &= \frac{1}{2}\|f\|_4^4 + \frac{1}{2}\|\tilde{f}\|_4^4 + \|f\tilde{f}\|_2^2 \\ &= \frac{1}{2}\|(|f(z)|^2 + |\tilde{f}(z)|^2)\|_2^2 \\ &\geq \frac{1}{2}\|(|f(z)|^2 + |\tilde{f}(z)|^2)\|_1^2 \\ &= \frac{1}{2}\left(\|f\|_2^2 + \|\tilde{f}\|_2^2\right)^2 \\ &= \frac{1}{2}\left(2\|f\|_2^2\right)^2 \\ &= 2\|f\|_2^4, \end{aligned}$$

where we use Lemma 2.2(ii) in the first and penultimate equalities, and the inequality uses the fact that the L^2 norm is always at least as large as the L^1 norm (by Jensen's inequality) because we are working on a space of measure 1. Therefore, $\left(\|f\|_4^4 + \|f\tilde{f}\|_2^2\right)/\|f\|_2^4 \geq 2$, which proves our lower bounds on the limiting values autocorrelation demerit factors. \square

Notice that although Theorem 2.7 has lower bounds on limiting autocorrelation demerit factors, no lower bound for the crosscorrelation demerit factor is given. This is because the only lower bound that could have been given is the trivial lower bound of 0, for the following example shows that one can obtain pairs of stems whose limiting crosscorrelation demerit factors are arbitrarily close to 0.

Proposition 2.8. *Let k be a positive integer, let $f_0(z) = (1 - z^{4k})/(1 - z)$, and $g_0(z) = (1 - z)(1 - z^2)(1 - z^{4k})/(1 - z^4)$. If $f_n(z)$ and $g_n(z)$ are defined recursively for all $n \in \mathbb{N}$ by*

$$\begin{aligned} f_{n+1}(z) &= f_n(z) + z^{\operatorname{len} f_n} f_n^\dagger(-z) \\ g_{n+1}(z) &= g_n(z) + z^{\operatorname{len} g_n} g_n^\dagger(-z), \end{aligned}$$

then

$$\lim_{n \rightarrow \infty} \operatorname{CDF}(f_n, g_n) = \frac{1}{3k}.$$

Remark 2.9. Note that f_0 in Proposition 2.8 is a Littlewood polynomial representing a sequence of length $4k$ whose terms are all 1. And g_0 is Littlewood polynomial representing a sequence of length $4k$ consisting of k repetitions of the smaller sequence $(1, -1, -1, 1)$.

Proposition 2.8 shows that we can find a pair of stems whose limiting crosscorrelation demerit factor is as close to 0 as we like simply by choosing a sufficiently high value of k when we define f_0 and g_0 .

In view of the Pursley-Sarwate bound (6), we know that the the autocorrelation performance for such stems cannot be exceptionally good, and in fact, one can use Theorem 2.7 to calculate the limiting autocorrelation demerit factors for the stems from seeds f_0 and g_0 described in Proposition 2.8. It is easy to calculate the sums of squares of the autocorrelation values for f_0 and for g_0 and also to compute the sums of squares of the crosscorrelation values for f_0 with \tilde{f}_0 and for g_0 with \tilde{g}_0 to show that $\|f_0\|_4^4 = 2k(32k^2 + 1)/3$, $\|g_0\|_4^4 = 4k(16k^2 + 5)/3$, $\|f_0\tilde{f}_0\|_2^2 = 4k$, and $\|g_0\tilde{g}_0\|_2^2 = 4k(16k^2 - 1)$. Since f_0 and g_0 are Littlewood polynomials of length $4k$, we have $\|f_0\|_2^2 = \|g_0\|_2^2 = 4k$, so that if f_0, f_1, \dots and g_0, g_1, \dots are the stems obtained from seeds f_0 and g_0 by recursion (1), then Theorem 2.7 tells us that $\lim_{n \rightarrow \infty} \text{ADF}(f_n) = \lim_{n \rightarrow \infty} \text{ADF}(g_n) = (16k^2 - 9k + 2)/(9k)$, which is strictly increasing from a value of 1 (when $k = 1$) to ∞ in the limit as $k \rightarrow \infty$.

Proof of Proposition 2.8. We shall use Theorem 2.7 to calculate the limiting crosscorrelation demerit factor. To that end, we calculate

$$\begin{aligned} f_0(z)g_0(z) &= \frac{(1 - z^{4k})^2(1 - z)(1 - z^2)}{(1 - z)(1 - z^4)} \\ &= \left(\frac{1 - z^{4k}}{1 - z^4} \right) (1 - z^2)(1 - z^{4k}) \\ &= (1 - z^2 + z^4 - z^6 + \dots + z^{4k-4} - z^{4k-2})(1 - z^{4k}), \end{aligned}$$

which is a polynomial with $4k$ nonzero coefficients, every of one of which is either 1 or -1 , so then $\|f_0g_0\|_2^2 = 4k$.

And then we calculate

$$\begin{aligned} f_0(z)\tilde{g}_0(z) &= \frac{(1 - z^{4k})^2(1 + z)(1 - z^2)}{(1 - z)(1 - z^4)} \\ &= \left(\frac{1 - z^{4k}}{1 - z^4} \right) (1 + z)^2(1 - z^{4k}) \\ &= (1 + 2z + z^2 + \dots + z^{4k-4} + 2z^{4k-3} + z^{4k-2})(1 - z^{4k}), \end{aligned}$$

which is a polynomial with $4k$ coefficients of magnitude 1 and $2k$ coefficients of magnitude 2, so then $\|f_0\tilde{g}_0\|_2^2 = 4k + 4 \cdot 2k = 12k$.

And then we calculate

$$\begin{aligned}
f_0(z)\widetilde{f_0}(z)\overline{g_0(z)\widetilde{g_0}(z)} &= \frac{(1-z^{4k})^2(1-z^{-4k})^2(1-z^{-2})^2(1-z^{-1})(1+z^{-1})}{(1-z)(1+z)(1-z^{-4})^2} \\
&= \frac{(1-z^{4k})^2(1-z^{-4k})^2(1-z^{-2})^2(-z^{-1})(z^{-1})}{(1-z^{-4})^2} \\
&= -\frac{(1-z^{4k})^2(1-z^{-4k})^2(1-z^{-2})^2z^2}{(1-z^{-4})^2z^4} \\
&= -\frac{(1-z^{4k})^2(1-z^{-4k})^2(1-z^2)(1-z^{-2})}{(1-z^4)(1-z^{-4})}.
\end{aligned}$$

Thus $\int f_0(z)\widetilde{f_0}(z)\overline{g_0(z)\widetilde{g_0}(z)} = -\|(1-z^{4k})^2(1-z^2)/(1-z^4)\|_2^2$, and we have already calculated the norm to be $4k$, and so $\int f_0(z)\widetilde{f_0}(z)\overline{g_0(z)\widetilde{g_0}(z)} = -4k$.

Finally, $\|f_0\|_2^2 = \|g_0\|_2^2 = 4k$ because f_0 and g_0 are Littlewood polynomials of length $4k$. Now Theorem 2.7 says that if $f_n(z)$ and $g_n(z)$ are defined recursively for all $n \in \mathbb{N}$ by

$$\begin{aligned}
f_{n+1}(z) &= f_n(z) + z^{\text{len } f_n} f_n^\dagger(-z) \\
g_{n+1}(z) &= g_n(z) + z^{\text{len } g_n} g_n^\dagger(-z),
\end{aligned}$$

then

$$\begin{aligned}
\lim_{n \rightarrow \infty} \text{CDF}(f_n, g_n) &= \frac{2\|f_0 g_0\|_2^2 + \|f_0 \widetilde{g_0}\|_2^2 + \text{Re} \int f_0 \widetilde{f_0} \overline{g_0 \widetilde{g_0}}}{3\|f_0\|_2^2 \|g_0\|_2^2} \\
&= \frac{2 \cdot 4k + 12k + \text{Re}(-4k)}{3(4k)^2} \\
&= \frac{1}{3k}. \quad \square
\end{aligned}$$

We close this section with the technical lemma used in the proof of Lemma 2.3 above.

Lemma 2.10. *If $a(z), b(z) \in \mathbb{C}[z, z^{-1}]$, $k \in \mathbb{Z}$, and $\sigma, \tau \in \{-1, 1\}$, and if*

$$I = \frac{1}{2} \left\| \begin{pmatrix} a + \sigma z^k \widetilde{a} \end{pmatrix} \begin{pmatrix} b + \tau z^k \widetilde{b} \end{pmatrix} \right\|_2^2 + \frac{1}{2} \left\| \begin{pmatrix} \widetilde{a} - \sigma z^k \overline{a} \end{pmatrix} \begin{pmatrix} \widetilde{b} - \tau z^k \overline{b} \end{pmatrix} \right\|_2^2,$$

then

$$I = 2\|ab\|_2^2 + 2\|a\widetilde{b}\|_2^2 + 2\sigma\tau \text{Re} \int a\widetilde{a}\overline{b\widetilde{b}} + 2\sigma\tau \text{Re} \int z^{-2k} a\widetilde{a}\overline{b\widetilde{b}}.$$

Proof. Note that

$$\begin{aligned}
I &= \frac{1}{2} \int \left(|a|^2 + |z^k \widetilde{a}|^2 + 2\sigma \text{Re} \left(a \overline{z^k \widetilde{a}} \right) \right) \left(|b|^2 + |z^k \widetilde{b}|^2 + 2\tau \text{Re} \left(b \overline{z^k \widetilde{b}} \right) \right) \\
&\quad + \frac{1}{2} \int \left(|\widetilde{a}|^2 + |z^k \overline{a}|^2 - 2\sigma \text{Re} \left(\widetilde{a} \overline{z^k a} \right) \right) \left(|\widetilde{b}|^2 + |z^k \overline{b}|^2 - 2\tau \text{Re} \left(\widetilde{b} \overline{z^k b} \right) \right),
\end{aligned}$$

and since we are integrating on the complex unit circle, we may omit terms of the form $|z^k|$ and replace \bar{z}^k with z^{-k} to obtain

$$I = \frac{1}{2} \int \left(|a|^2 + |\tilde{a}|^2 + 2\sigma \operatorname{Re} \left(z^{-k} a \tilde{a} \right) \right) \left(|b|^2 + |\tilde{b}|^2 + 2\tau \operatorname{Re} \left(z^{-k} b \tilde{b} \right) \right) \\ + \frac{1}{2} \int \left(|\tilde{a}|^2 + |a|^2 - 2\sigma \operatorname{Re} \left(z^{-k} \tilde{a} a \right) \right) \left(|\tilde{b}|^2 + |b|^2 - 2\tau \operatorname{Re} \left(z^{-k} \tilde{b} b \right) \right),$$

from which one obtains

$$I = \int (|a|^2 + |\tilde{a}|^2) (|b|^2 + |\tilde{b}|^2) + 4\sigma\tau \int \operatorname{Re} \left(z^{-k} a \tilde{a} \right) \operatorname{Re} \left(z^{-k} b \tilde{b} \right) \\ = \|ab\|_2^2 + \|\tilde{a}\tilde{b}\|_2^2 + \|a\tilde{b}\|_2^2 + \|\tilde{a}b\|_2^2 + 4\sigma\tau \int \operatorname{Re} \left(z^{-k} a \tilde{a} \right) \operatorname{Re} \left(z^{-k} b \tilde{b} \right) \\ = 2\|ab\|_2^2 + 2\|\tilde{a}\tilde{b}\|_2^2 + 4\sigma\tau \int \operatorname{Re} \left(z^{-k} a \tilde{a} \right) \operatorname{Re} \left(z^{-k} b \tilde{b} \right) \\ = 2\|ab\|_2^2 + 2\|\tilde{a}\tilde{b}\|_2^2 + 2\sigma\tau \int \left[\operatorname{Re} \left(z^{-k} a \tilde{a} z^{-k} b \tilde{b} \right) + \operatorname{Re} \left(z^{-k} a \tilde{a} \overline{z^{-k} b \tilde{b}} \right) \right],$$

where the third equality uses Lemma 2.2(ii) and the fourth equality uses the observation that $2 \operatorname{Re}(u) \operatorname{Re}(v) = \operatorname{Re}(uv) + \operatorname{Re}(u\bar{v})$. The desired result now readily follows. \square

3. SYMMETRY GROUPS

The expressions in Theorem 1.2 for the limiting autocorrelation and cross-correlation demerit factors are invariant under certain symmetries. This helps abbreviate computational searches for sequences and sequence pairs with optimum performance. These symmetries are based negation of polynomials, replacement of z by $-z$ in polynomials, and transformation of polynomials to their conjugate reciprocals. One should recall the notational conventions $\tilde{f}(z)$ and $\overline{f(z)}$ for $f(z) \in \mathbb{C}[z, z^{-1}]$ and the definition of the conjugate reciprocal $f^\dagger(z)$ for $f(z) \in \mathbb{C}[z]$ from the first paragraph of Section 2. One should note that $\tilde{f}g(z) = \tilde{f}(z)\tilde{g}(z)$ and $\overline{f(z)g(z)} = \overline{f(z)} \cdot \overline{g(z)}$ for every $f(z), g(z) \in \mathbb{C}[z, z^{-1}]$ and $(f(z)g(z))^\dagger = f^\dagger(z)g^\dagger(z)$ for every $f(z), g(z) \in \mathbb{C}[z]$. (The third relation follows easily from Lemma 2.1(i) and the second relation.) We shall also need the following observation.

Lemma 3.1. *For any $f(z) \in \mathbb{C}[z]$ and any $p \in \mathbb{R}$ with $p \geq 1$, we have $\|f^\dagger(z)\|_p = \|f(z)\|_p$.*

Proof. By Lemma 2.1(ii), we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f^\dagger(e^{i\theta})|^p d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| (e^{i\theta})^{\deg f} \overline{f(e^{i\theta})} \right|^p d\theta \\ = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta. \quad \square$$

Now we introduce a group of symmetries that, when applied to a polynomial $f_0(z) \in \mathbb{C}[z]$ with nonzero constant coefficient, will preserve the autocorrelation properties of the stem f_0, f_1, \dots of Rudin-Shapiro-like polynomials obtained from seed f_0 via recursion (1). First we describe our group and how it affects certain norms, and then we show its effect on autocorrelation as a corollary.

Proposition 3.2. *Let ℓ be a nonnegative integer, and let P_ℓ be the set of all polynomials of length ℓ in $\mathbb{C}[z]$ that have nonzero constant coefficient. We define three maps from P_ℓ to itself:*

$$\begin{aligned} n(f) &= -f \\ h(f) &= \widetilde{f} \\ r(f) &= f^\dagger. \end{aligned}$$

These maps generate a group $G_\ell = \langle n, h, r \rangle$ of permutations of P_ℓ .

- (i). If $\ell = 1$, then G_1 is the internal direct product of the two cyclic groups $\langle n \rangle$ and $\langle r \rangle$, each of order 2.
- (ii). If ℓ is odd with $\ell > 1$, then G_ℓ is the internal direct product of the three cyclic groups $\langle n \rangle$, $\langle h \rangle$, and $\langle r \rangle$, each of order 2.
- (iii). If ℓ is even, then G_ℓ is isomorphic to the dihedral group of order 8 generated by rh and h , where rh is of order 4, h is of order 2, and $h(rh)h^{-1} = (rh)^{-1}$.

For any $t \in G_\ell$, any $f \in P_\ell$, and any $p \geq 1$, we have

$$\begin{aligned} \|t(f)\|_p &= \|f\|_p \\ \|t(f)t(\widetilde{f})\|_p &= \|f\widetilde{f}\|_p. \end{aligned}$$

Proof. It is clear that each of n , h , and r is an involution on P_ℓ (except that h is the identity element when $\ell = 1$), so these maps generate a group of permutations of P_ℓ . If $\ell = 1$, then it is not hard to show that $e^{\pi i/4} \in P_1$ that has four distinct images under $G_\ell = \langle n, h, r \rangle$, so G_ℓ has order at least 4. If $\ell > 1$, then it is not hard to show that $e^{\pi i/6} + e^{\pi i/3}z^{\ell-1} \in P_\ell$ has eight distinct images under $G_\ell = \langle n, h, r \rangle$, so G_ℓ has order at least 8. Furthermore n commutes with both h and r , and Lemma 2.1(ii) shows that $hr = rh$ when ℓ is odd, but $hr = nrh$ when ℓ is even.

Thus if $\ell = 1$, then $G_1 = \langle n, h, r \rangle = \langle n, r \rangle$ is a group of order at least 4, generated by commuting involutions n and r . So G_1 is the internal direct product of $\langle n \rangle$ and $\langle r \rangle$, which are both cyclic groups of order 2.

If ℓ is odd and greater than 1, our group $G_\ell = \langle n, h, r \rangle$ is a group of order at least 8 generated by commuting involutions n , h , and r . So G_ℓ is the internal direct product of $\langle n \rangle$, $\langle h \rangle$, and $\langle r \rangle$, which are three cyclic groups each of order 2.

On the other hand, if ℓ is even, then rh can be shown to have $(rh)^2 = x^2$, $(rh)^3 = x^2rh = hr$, and $(rh)^4$ the identity, and since the powers of rh take the element $1 + iz^{\ell-1} \in P_\ell$ to four distinct elements, we see that rh has

order 4. Thus $G_\ell = \langle n, h, r \rangle = \langle h, r \rangle = \langle h, rh \rangle$. Then note that $h(rh)h^{-1} = hr = (rh)^3 = (rh)^{-1}$, and so it can be seen that $G_\ell = \langle h, rh \rangle$ is generated by an element h of order 2 and an element $y = rh$ of order 4 that satisfy the relation $hyh^{-1} = y^{-1}$. These are the relations satisfied by the generators of the dihedral group D of order 8, that is, the group of symmetries of a square (with a 90° rotation corresponding to y and a flip corresponding to h). So G_ℓ is a homomorphic image of D , but since G_ℓ has order at least 8, we must have $G_\ell \cong D$.

To verify that $\|t(f)\|_p = \|f\|_p$ for any $f \in P_\ell$, $t \in G_\ell$, and $p \geq 1$, it suffices to check that it holds when t is one of the generators n , h , and r . When $t = n$, this is clear, and when $t = h$ or r , it is a consequence, respectively, of Lemma 2.2(ii) or Lemma 3.1.

Similarly, to verify that $\|t(f)\widetilde{t(f)}\|_p = \|f\widetilde{f}\|_p$ for any $f \in P_\ell$, $t \in G_\ell$, and $p \geq 1$, it suffices to check that it holds when t is one of the generators n , h , and r . When $t = n$ or h , this is clear, and when $t = r$, then

$$\begin{aligned} \|r(f)\widetilde{r(f)}\|_p &= \|f_0^\dagger(\widetilde{f_0^\dagger})\|_p \\ &= \|f_0^\dagger(\widetilde{f_0})^\dagger\|_p \text{ or } \|-f_0^\dagger(\widetilde{f_0})^\dagger\|_p \\ &= \|(f_0\widetilde{f_0})^\dagger\|_p \\ &= \|f_0\widetilde{f_0}\|_p, \end{aligned}$$

where the second equality uses Lemma 2.1(ii), and the fourth equality uses Lemma 3.1. \square

Corollary 3.3. *Let $f_0 \in \mathbb{C}[z]$ be a polynomial of length ℓ with nonzero constant coefficient, let t be an element of the group G_ℓ be the group described in Proposition 3.2, and let $a_0 = t(f_0)$. If f_0, f_1, \dots and a_0, a_1, \dots are sequences of Rudin-Shapiro-like polynomials generated from f_0 and a_0 via recursion (1), then*

$$\lim_{n \rightarrow \infty} \text{ADF}(a_n) = \lim_{n \rightarrow \infty} \text{ADF}(f_n).$$

Proof. By Theorem 1.2, we have

$$\lim_{n \rightarrow \infty} \text{ADF}(f_n) = -1 + \frac{2}{3} \cdot \frac{\|f_0\|_4^4 + \|f_0\widetilde{f_0}\|_2^2}{\|f_0\|_2^4},$$

but Proposition 3.2 shows that the values of the three norms occurring on the right hand side do not change if we replace every instance of f_0 with $t(f_0) = a_0$, which changes the right hand side to $\lim_{n \rightarrow \infty} \text{ADF}(a_n)$ by Theorem 1.2. \square

Now we introduce a group of symmetries that, when applied to a pair of polynomials $(f_0(z), g_0(z))$ from $\mathbb{C}[z]$, will preserve the crosscorrelation properties of the stems f_0, f_1, \dots and g_0, g_1, \dots of Rudin-Shapiro-like polynomials obtained from seeds f_0 and g_0 via recursion (1). First we describe the group and how it affects certain norms and integrals, and then we show its effect on crosscorrelation as a corollary.

Proposition 3.4. *Let ℓ be a nonnegative integer, and let P_ℓ be the set of all polynomials of length ℓ in $\mathbb{C}[z]$ that have nonzero constant coefficient. We define four maps from $P_\ell \times P_\ell$ to itself:*

$$\begin{aligned} s(f, g) &= (g, f) \\ n(f, g) &= (-f, g) \\ h(f, g) &= (\tilde{f}, \tilde{g}) \\ r(f, g) &= (f^\dagger, g^\dagger). \end{aligned}$$

These maps generate a group $G_{\ell, \ell} = \langle s, n, h, r \rangle$ of permutations of $P_\ell \times P_\ell$. $G_{\ell, \ell}$ contains a dihedral subgroup D of order 8 generated ns and s , where ns has order 4, s has order 2, and $s(ns)s^{-1} = (ns)^{-1}$.

- (i). If $\ell = 1$, then $G_{\ell, \ell}$ is the internal direct product of the dihedral group D of order 8 and the cyclic group $\langle r \rangle$ of order 2.
- (ii). If ℓ is odd and $\ell > 1$, $G_{\ell, \ell}$ is the internal direct product of the dihedral group D of order 8, the cyclic subgroup $\langle h \rangle$ of order 2, and the cyclic subgroup $\langle r \rangle$ of order 2.
- (iii). If ℓ is even, then $G_{\ell, \ell}$ is the internal central product of D and another dihedral subgroup Δ of order 8 generated by rh and h , where rh has order 4, h has order 2, and $h(rh)h^{-1} = (rh)^{-1}$. Thus $G_{\ell, \ell}$ is isomorphic to the extraspecial group of order 2^5 of $+$ type, which is also the inner holomorph of the dihedral group of order 8.

For any $t \in G_{\ell, \ell}$, any $f, g \in P_\ell$, and any $p \geq 1$, we have

$$\begin{aligned} \|t(f)\|_p \|t(g)\|_p &= \|f\|_p \|g\|_p \\ \|t(f)t(g)\|_p &= \|fg\|_p \\ \|t(f)\widetilde{t(g)}\|_p &= \|f\widetilde{g}\|_p \\ \operatorname{Re} \int t(f)\widetilde{t(f)}\overline{t(g)t(g)} &= \operatorname{Re} \int f\widetilde{f}\overline{g\widetilde{g}}. \end{aligned}$$

Proof. It is clear that each of s , n , h , and r is an involution on $P_\ell \times P_\ell$ (except that h is the identity element when $\ell = 1$), so these maps generate a group of permutations of $P_\ell \times P_\ell$. Furthermore, s commutes with h and r , and n also commutes with h and r . Thus we can better understand our group $G_{\ell, \ell}$ by focusing on two subgroups, $\langle s, n \rangle$ and $\langle h, r \rangle$, with the knowledge that every element from the former subgroup commutes with every element of the latter subgroup.

Let us first focus on the subgroup $\langle s, n \rangle$ of $G_{\ell, \ell}$. We note that $(sn)(f, g) = (g, -f)$ but $(ns)(f, g) = (-g, f)$, so that s and n do not commute. We define $x = ns$, and then $\langle s, n \rangle = \langle s, ns \rangle = \langle s, x \rangle$. We note that x is an element of order 4 with $x^2(f, g) = (-f, -g) = -(f, g)$ and $x^3(f, g) = (g, -f) = (sn)(f, g)$. Then note that $sxs^{-1} = snss^{-1} = sn = x^3 = x^{-1}$. Thus $\langle s, n \rangle = \langle s, x \rangle$ must be a homomorphic image of a dihedral group of order 8, the group of symmetries of a square (with a 90° rotation corresponding

to x and a flip corresponding to s). And in fact, one can show that the $(1 + z^{\ell-1}, i + iz^{\ell-1}) \in P_\ell \times P_\ell$ has eight distinct images under the action of $\langle s, n \rangle$, so $\langle s, x \rangle \cong D$.

Now let us focus on the subgroup $\langle h, r \rangle$ of $G_{\ell, \ell}$. If $\ell = 1$, then h is the identity element, so $\langle h, r \rangle = \langle r \rangle$ is a cyclic group of order 2. Since the elements of $\langle s, n \rangle$ commute with the elements of $\langle h, r \rangle$, this means that $G_{1,1}$ is a homomorphic image of a direct product of a dihedral group D of order 8 and a cyclic group C of order 2. If $\ell = 1$, then it is not hard to show that $(1, e^{\pi i/4}) \in P_1 \times P_1$ that has 16 distinct images under $G_{1,1} = \langle n, h, r \rangle$, so $G_{1,1} \cong D \times C$.

Now suppose that $\ell > 1$. Lemma 2.1(ii) shows that $hr = rh$ when ℓ is odd, but that $(hr)(f, g) = -(rh)(f, g)$ when ℓ is even. Note that the group element which maps (f, g) to $(-f, -g)$ is x^2 described in the previous paragraph. So h and r commute when ℓ is odd, but $hr = x^2 rh$ when ℓ is even. So if ℓ is odd, then $\langle h, r \rangle$ is a homomorphic image of a Klein four-group, that is, of $C \times C$ with C a cyclic group of order 2. And in fact one can show that $(1 + z - z^{\ell-1}, 1 + z - z^{\ell-1})$ has four distinct images under the action of $\langle h, r \rangle$, so $\langle h, r \rangle \cong C \times C$.

On the other hand, if ℓ is even, then rh can be shown to have $(rh)^2 = x^2$, $(rh)^3 = x^2 rh = hr$, and $(rh)^4$ the identity, and since the powers of rh take the element $(1 + iz^{\ell-1}, 1 + iz^{\ell-1}) \in P_\ell \times P_\ell$ to four distinct elements, we see that rh has order 4. Note that $h(rh)h^{-1} = hr = (rh)^3 = (rh)^{-1}$, and so it can be seen that $\langle h, r \rangle = \langle h, rh \rangle$ is generated by an element h of order 2 and an element $y = rh$ of order 4 that satisfy the relation $hyh^{-1} = y^{-1}$. Thus if ℓ is even, then $\langle r, h \rangle$ is a homomorphic image of the dihedral group D of order 8, the group of symmetries of a square (with a 90° rotation corresponding to y and a flip corresponding to h). One can show that $(1 + iz^{\ell-1}, 1 + iz^{\ell-1})$ has eight distinct images under the action of $\langle h, r \rangle$, so that $\langle h, r \rangle \cong D$.

Now we assemble what we have learned about the subgroups $\langle s, n \rangle$ and $\langle h, r \rangle$ of $G_{\ell, \ell}$ using the fact that every element in the former subgroup commutes with every element in the latter. When $\ell > 1$, then it is not hard to show that $(e^{\pi i/6} + e^{\pi i/3} z^{\ell-1}, 1 + z^{\ell-1}) \in P_\ell \times P_\ell$ has 32 distinct images under $G_{\ell, \ell} = \langle n, h, r \rangle$, so $G_{\ell, \ell}$ has order at least 32. We saw that $\langle s, n \rangle$ is a dihedral group of order 8 generated by element $x = ns$ of order 4 and element s of order 2. This group has a center $\langle x^2 \rangle$ of order 2.

If ℓ is odd and greater than 1, we saw that $\langle h, r \rangle$ is a Klein four-group generated by the elements h and r , each of order 2. So then the group $G_{\ell, \ell} = \langle s, n, h, r \rangle$ is a homomorphic image of $D \times C \times C$, where D is the dihedral group of order 8 and C is the cyclic group of order 2. But since $G_{\ell, \ell}$ has at least 32 elements, we must have $G_{\ell, \ell} \cong D \times C \times C$.

On the other hand, if ℓ is even, then $\langle h, r \rangle$ is a dihedral group of order 8 generated by element $y = rh$ of order 4 and element h of order 2. This group has a center $\langle y^2 \rangle$, and we observed above that $y^2 = (rh)^2 = x^2$, so the centers of $\langle n, s \rangle$ and $\langle h, r \rangle$ completely overlap with each other. So the group $G_{\ell, \ell} = \langle s, n, h, r \rangle$ is a homomorphic image of the central product of

two dihedral groups of order 8, which makes $G_{\ell,\ell}$ a homomorphic image of the extraspecial group of order 2^5 of + type. But since $G_{\ell,\ell}$ has at least 32 elements, $G_{\ell,\ell}$ must be isomorphic to the extraspecial group of order 2^5 of + type, which is also the inner holomorph of the dihedral group of order 8.

Now we verify that the four invariance relations for $G_{\ell,\ell}$ in the statement of this proposition. It suffices to check these relations when the group element $t \in G_{\ell,\ell}$ is one of the four generators s, n, h, r of the group.

For any $(f, g) \in P_\ell \times P_\ell$ and $p \geq 1$, it is clear that $\|t(f)\|_p \|t(g)\|_p = \|f\|_p \|g\|_p$ when $t = s$ or n , and when $t = h$ or r , this is a consequence, respectively, of Lemma 2.2(ii) or Lemma 3.1.

For any $(f, g) \in P_\ell \times P_\ell$ and $p \geq 1$, it is clear that $\|t(f)t(g)\|_p = \|fg\|_p$ when $t = s$ or n , and when $t = h$ or r , then this is a consequence, respectively, of Lemma 2.2(ii) or Lemma 3.1.

Now we verify that $\|t(f)t(g)\|_p = \|f\widetilde{g}\|_p$ for any $(f, g) \in P_\ell \times P_\ell$, $p \geq 1$, and $t \in \{s, n, h, r\}$. This is clear when $t = n$, and $\|s(f)s(g)\|_p = \|h(f)h(g)\|_p = \|\widetilde{f}g\|_p$, which equals $\|f\widetilde{g}\|_p$ by Lemma 2.2(ii). When $t = r$, then

$$\begin{aligned} \|r(f)r(g)\|_p &= \|f^\dagger(\widetilde{g^\dagger})\|_p \\ &= \|f^\dagger(\widetilde{g})^\dagger\|_p \text{ or } \|-f^\dagger(\widetilde{g})^\dagger\|_p \\ &= \|(f\widetilde{g})^\dagger\|_p \\ &= \|f\widetilde{g}\|_p, \end{aligned}$$

where Lemmata 2.1(ii) and 3.1 are used in the second and fourth equalities.

Now we verify that $\operatorname{Re} \int t(f)\overline{t(f)}t(g)\overline{t(g)} = \operatorname{Re} \int f\widetilde{f}g\widetilde{g}$ for any $(f, g) \in P_\ell \times P_\ell$, $p \geq 1$, and $t \in \{s, n, h, r\}$. When $t = n$ or h , this is clear, and when $t = s$, we see that $\operatorname{Re} \int s(f)\overline{s(f)}s(g)\overline{s(g)} = \operatorname{Re} \int f\widetilde{f}g\widetilde{g}$, which is equal to $\operatorname{Re} \int f\widetilde{f}g\widetilde{g}$, since conjugation of the integrand does not change the real part of the integral. Finally, if $t = r$, then

$$\begin{aligned} \operatorname{Re} \int r(f)\overline{r(f)}r(g)\overline{r(g)} &= \operatorname{Re} \int f^\dagger\widetilde{f^\dagger}g^\dagger\overline{g^\dagger} \\ &= \operatorname{Re} \int f^\dagger(\widetilde{f})^\dagger\overline{g^\dagger(\widetilde{g})^\dagger} \\ &= \operatorname{Re} \int \overline{(f\widetilde{f})^\dagger(g\widetilde{g})^\dagger} \\ &= \operatorname{Re} \int \left(\overline{z^{-2\deg f}f\widetilde{f}}\right) \left(z^{-2\deg g}g\widetilde{g}\right) \\ &= \operatorname{Re} \int \overline{f\widetilde{f}}g\widetilde{g} \\ &= \operatorname{Re} \int f\widetilde{f}g\widetilde{g}, \end{aligned}$$

where the second equality uses Lemma 2.1(ii) (and the fact that f and g are assumed to have the same degree), the fourth equality uses Lemma 2.1(iii), the fifth equality uses the fact that $\bar{z} = z^{-1}$ on the complex unit circle, and the last equality uses the fact that conjugation of the integrand does not change the real part of the integral. \square

Corollary 3.5. *Let $f_0, g_0 \in \mathbb{C}[z]$ be a polynomials of length ℓ with nonzero constant coefficients, let t be an element of the group $G_{\ell, \ell}$ be the group described in Proposition 3.4, and let $(a_0, b_0) = t(f_0, g_0)$. If f_0, f_1, \dots and g_0, g_1, \dots and a_0, a_1, \dots and b_0, b_1, \dots are sequences of Rudin-Shapiro-like polynomials generated from f_0, g_0, a_0 , and b_0 via recursion (1), then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{CDF}(a_n, b_n) &= \lim_{n \rightarrow \infty} \text{CDF}(f_n, g_n) \text{ and} \\ \lim_{n \rightarrow \infty} \text{PSC}(a_n, b_n) &= \lim_{n \rightarrow \infty} \text{PSC}(f_n, g_n). \end{aligned}$$

Proof. By Theorem 1.2, we have

$$\lim_{n \rightarrow \infty} \text{CDF}(f_n, g_n) = \frac{2\|f_0 g_0\|_2^2 + \|f_0 \tilde{g}_0\|_2^2 + \text{Re} \int f_0 \tilde{f}_0 \overline{g_0 \tilde{g}_0}}{3\|f_0\|_2^2 \|g_0\|_2^2},$$

but Proposition 3.4 shows that the values of the three terms in the numerator and the value of the denominator of the right hand side do not change if we replace every instance of f_0 with $t(f_0) = a_0$ and every instance of g_0 with $t(g_0) = b_0$. These replacements change the right hand side to $\lim_{n \rightarrow \infty} \text{CDF}(a_n, b_n)$ by Theorem 1.2.

Because of the structure of the groups G_ℓ and $G_{\ell, \ell}$ described in Propositions 3.2 and 3.4 above, one can say that there exist $u, v \in G_\ell$ such that either $(a_0, b_0) = (u(f_0), v(g_0))$ or $(a_0, b_0) = (v(g_0), u(f_0))$. Thus by Corollary 3.3,

$$\lim_{n \rightarrow \infty} \text{ADF}(a_n) \text{ADF}(b_n) = \lim_{n \rightarrow \infty} \text{ADF}(f_n) \text{ADF}(g_n),$$

and so, considering the formula (7) for the Pursley-Sarwate criterion, we see that

$$\lim_{n \rightarrow \infty} \text{PSC}(a_n, b_n) = \lim_{n \rightarrow \infty} \text{PSC}(f_n, g_n). \quad \square$$

4. SOME EXAMPLES OF PAIRS OF RUDIN-SHAPIRO-LIKE SEQUENCES WITH LOW CORRELATION

For each $\ell \leq 40$, we considered every possible Littlewood polynomial f_0 of length ℓ , and programmed a computer to calculate via Corollary 2.5 (a result originally due to Borwein and Mossinghoff [2, Theorem 1]) the limiting autocorrelation demerit factor of the stem f_0, f_1, \dots constructed from our recursion (1). For each length ℓ , we report on Table 1 the lowest limiting autocorrelation demerit factor achieved, and indicate how many seeds achieve this minimum value. Seeds that are equivalent modulo the action of the group G_ℓ described in Proposition 3.2 always have the same limiting autocorrelation demerit factor by Corollary 3.3, and so we group seeds into orbits under the action of G_ℓ and report how many distinct orbits

there are on Table 1. For each length ℓ , we give one example of a seed f_0 whose stem achieves the smallest limiting autocorrelation demerit factor.

Example seeds are reported using a hexadecimal code. To decode, expand each hexadecimal digit into binary form ($0 \rightarrow 0000$, $1 \rightarrow 0001$, \dots , $F \rightarrow 1111$) and, if necessary, remove initial 0 symbols to obtain a binary sequence of the appropriate length. Then convert each 0 to +1 and each 1 to -1 to obtain the list of coefficients of the seed f_0 . For example, Table 2 reports for length $\ell = 14$ that one seed of interest is 149B. Expand to 0001 0100 1001 1011 and delete the initial two zeroes to obtain a sequence 01 0100 1001 1011 of length $\ell = 14$. Convert from 0, 1 to ± 1 to obtain the coefficients of

$$g_0(z) = 1 - z + z^2 - z^3 + z^4 + z^5 - z^6 + z^7 + z^8 - z^9 - z^{10} + z^{11} - z^{12} - z^{13}.$$

Borwein-Mossinghoff [2, Corollary 1] proved that the limiting autocorrelation demerit factor for any sequence f_0, f_1, \dots of Rudin-Shapiro-like Littlewood polynomials generated from a seed f_0 using recursion (1) can never be less than $1/3$ (see also our Theorem 1.2). Their computer experiments show that for seeds of length $\ell \leq 40$, there exist seeds whose stems achieve limiting autocorrelation demerit factor $1/3$ only when $\ell \in \{1, 2, 4, 8, 16, 20, 32, 40\}$. They also indicate how many seeds of each of these lengths give stems with limiting autocorrelation demerit factor $1/3$. Our computer experiments agree with theirs, but we also present on Table 1 the minimum limiting autocorrelation demerit factors for all lengths $\ell \leq 40$, regardless of whether or not the minimum is $1/3$.

TABLE 1. Lowest Limiting Autocorrelation Demerit Factor for Seeds of Each Length

seed length	limiting ADF(f_n)	number of sequences	number of orbits	sample f_0
1	$\frac{1}{3} = 0.3333 \dots$	2	1	0
2	$\frac{1}{3} = 0.3333 \dots$	4	1	0
3	$\frac{17}{27} = 0.6296 \dots$	8	2	0
4	$\frac{1}{3} = 0.3333 \dots$	8	1	1
5	$\frac{41}{75} = 0.5466 \dots$	24	4	01
6	$\frac{17}{27} = 0.6296 \dots$	56	8	01
7	$\frac{73}{147} = 0.4965 \dots$	56	8	03
8	$\frac{1}{3} = 0.3333 \dots$	32	4	06
9	$\frac{113}{243} = 0.4650 \dots$	144	18	006
10	$\frac{41}{75} = 0.5466 \dots$	504	64	006
11	$\frac{161}{363} = 0.4435 \dots$	168	22	01C

TABLE 1. (continued) Lowest Limiting Autocorrelation Demerit Factor for Seeds of Each Length

seed length	limiting ADF(f_n)	number of sequences	number of orbits	sample f_0
12	$\frac{11}{27} = 0.4074 \dots$	96	12	036
13	$\frac{217}{507} = 0.4280 \dots$	344	44	0036
14	$\frac{73}{147} = 0.4965 \dots$	2648	332	0036
15	$\frac{281}{675} = 0.4162 \dots$	688	86	0163
16	$\frac{1}{3} = 0.3333 \dots$	192	24	0359
17	$\frac{353}{867} = 0.4071 \dots$	1472	184	001C9
18	$\frac{113}{243} = 0.4650 \dots$	12992	1624	001C9
19	$\frac{433}{1083} = 0.3998 \dots$	784	98	00793
20	$\frac{1}{3} = 0.3333 \dots$	128	16	05239
21	$\frac{521}{1323} = 0.3938 \dots$	1312	164	000F19
22	$\frac{161}{363} = 0.4435 \dots$	35352	4420	000F19
23	$\frac{617}{1587} = 0.3887 \dots$	1696	212	0066B4
24	$\frac{19}{54} = 0.3518 \dots$	320	40	00CD69
25	$\frac{721}{1875} = 0.3845 \dots$	2176	272	000CD29
26	$\frac{217}{507} = 0.4280 \dots$	104920	13116	0007866
27	$\frac{833}{2187} = 0.3808 \dots$	1888	236	006B274
28	$\frac{53}{147} = 0.3605 \dots$	512	64	00DB171
29	$\frac{953}{2523} = 0.3777 \dots$	3040	380	000E4B8D
30	$\frac{281}{675} = 0.4162 \dots$	266688	33336	0006C729
31	$\frac{1081}{2883} = 0.3749 \dots$	6368	796	001E2D33
32	$\frac{1}{3} = 0.3333 \dots$	1536	192	003C5A66
33	$\frac{1217}{3267} = 0.3725 \dots$	10400	1300	0003C5A66
34	$\frac{353}{867} = 0.4071 \dots$	554752	69344	0003C5A66
35	$\frac{1361}{3675} = 0.3703 \dots$	1216	152	001F1699C
36	$\frac{29}{81} = 0.3580 \dots$	640	80	0034EC5A6
37	$\frac{1513}{4107} = 0.3683 \dots$	1760	220	00035AC726
38	$\frac{433}{1083} = 0.3998 \dots$	840256	105032	00034E94E6
39	$\frac{1673}{4563} = 0.3666 \dots$	4416	552	0019E2D2B3
40	$\frac{1}{3} = 0.3333 \dots$	1088	136	0033C5A566

Now let us also consider crosscorrelation. In view of Proposition 2.8 and Remark 2.9, we already know we can achieve limiting crosscorrelation demerit factors as close to 0 as we like, but we have observed that pairs of stems with very low limiting crosscorrelation demerit factor tend to have very high autocorrelation demerit factor, and are therefore of little practical value. This is not surprising, given the bound (8) of Pursley and Sarwate. It is much more enlightening to ask how low one can make the limiting Pursley-Sarwate criterion (7), which combines both autocorrelation and crosscorrelation performance.

For each $\ell \leq 20$, we considered every possible pair of Littlewood polynomials (f_0, g_0) of length ℓ , and programmed a computer to calculate via Theorem 1.2 the limiting crosscorrelation demerit factors of the pair of stems $(f_0, f_1, \dots; g_0, g_1, \dots)$ constructed from our recursion (1). We also calculate the limiting autocorrelation demerit factors for each of the two stems, and from all three of these limits, we also obtain the limiting Pursley-Sarwate criterion. Table 2 records the lowest limiting Pursley-Sarwate criterion achieved for each $\ell \leq 20$, and records the seed pairs (f_0, g_0) that give rise to the pairs of stems that achieve this minimum. Seed pairs that are equivalent modulo the action of the group $G_{\ell, \ell}$ described in Proposition 3.4 will always have the same limiting Pursley-Sarwate criterion by Corollary 3.5, and so we group seeds pairs into orbits under the action of $G_{\ell, \ell}$. We report one representative of each class on Table 2 using our hexadecimal code (described above in the discussion of Table 1) and also report the size of the orbit. For some lengths there are multiple equivalence classes that achieve the same minimum limiting Pursley-Sarwate criterion: each such class has its own line on the table.

TABLE 2. Lowest Limiting Pursley-Sarwate Criterion for Seeds of Each Length

seed length	limiting values as $n \rightarrow \infty$				orbit size	seeds	
	$\text{PSC}(f_n, g_n)$	$\text{ADF}(f_n)$	$\text{ADF}(g_n)$	$\text{CDF}(f_n, g_n)$		f_0	g_0
1	1.6666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	4	0	0
2	1.3333...	$\frac{1}{3}$	$\frac{1}{3}$	1	8	0	1
3	1.3703...	$\frac{17}{27}$	$\frac{17}{27}$	$\frac{20}{27}$	32	0	1
4	1.1666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	16	1	2
5	1.3466...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{4}{5}$	32	01	02
5	1.3466...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{4}{5}$	32	01	08
5	1.3466...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{4}{5}$	32	01	0D
6	1.2962...	$\frac{17}{27}$	$\frac{17}{27}$	$\frac{2}{3}$	32	02	0D
6	1.2962...	$\frac{17}{27}$	$\frac{17}{27}$	$\frac{2}{3}$	32	04	0B
7	1.2312...	$\frac{73}{147}$	$\frac{73}{147}$	$\frac{36}{49}$	32	04	1A
8	1.1666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	32	06	3A
8	1.1666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	32	12	2E
9	1.2057...	$\frac{113}{243}$	$\frac{113}{243}$	$\frac{20}{27}$	32	009	035
10	1.1733...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{47}{75}$	32	04D	0A1
11	1.1818...	$\frac{161}{363}$	$\frac{161}{363}$	$\frac{268}{363}$	32	032	251
12	1.1666...	$\frac{11}{27}$	$\frac{11}{27}$	$\frac{41}{54}$	32	065	6A3
13	1.1734...	$\frac{217}{507}$	$\frac{281}{507}$	$\frac{116}{169}$	32	00CA	03AD
13	1.1734...	$\frac{217}{507}$	$\frac{281}{507}$	$\frac{116}{169}$	32	00CA	0907
14	1.1836...	$\frac{73}{147}$	$\frac{73}{147}$	$\frac{101}{147}$	32	0071	149B
15	1.1546...	$\frac{281}{675}$	$\frac{23}{45}$	$\frac{52}{75}$	32	024E	15C3
16	1.1041...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{37}{48}$	32	0A36	11D2
17	1.1407...	$\frac{353}{867}$	$\frac{353}{867}$	$\frac{212}{289}$	32	0038D	0EE96
18	1.1481...	$\frac{113}{243}$	$\frac{113}{243}$	$\frac{166}{243}$	32	0039A	0E8F6
19	1.1559...	$\frac{433}{1083}$	$\frac{497}{1083}$	$\frac{788}{1083}$	32	00E4D	38A16
19	1.1559...	$\frac{433}{1083}$	$\frac{497}{1083}$	$\frac{788}{1083}$	32	0C56D	0E013
20	1.1363...	$\frac{1}{3}$	$\frac{11}{25}$	$\frac{113}{150}$	32	08FA6	5A230

In Table 3 we also present sequences with low limiting Pursley-Sarwate criterion. For a given length $\ell \leq 40$, we first found all the seeds that produce stems whose autocorrelation demerit factors reach the minimum value for

that length, as reported in Table 1. Then we compute which pairs of these seeds (f_0, g_0) produce pairs of stems with the lowest limiting crosscorrelation demerit factor (and therefore the lowest limiting Pursley-Sarwate criterion, since they all have the same limiting autocorrelation demerit factors). Seed pairs that are equivalent modulo the action of the group $G_{\ell, \ell}$ described in Proposition 3.4 will always have the same limiting Pursley-Sarwate criterion by Corollary 3.5, and so we group seeds pairs into orbits under the action of $G_{\ell, \ell}$. We report one representative of each class on Table 3 using our hexadecimal code (described above in the discussion of Table 1) and also report the size of the orbit. For some lengths there are multiple equivalence classes that achieve the same minimum limiting Pursley-Sarwate criterion: each such class has its own line on the table.

One can compare the results on Table 2 (which reports the minimum limiting Pursley-Sarwate criterion over all pairs of stems of a given length) with the results of Table 3 (which reports the minimum limiting Pursley-Sarwate criterion only over pairs of stems whose limiting autocorrelation demerit factors equal the minimum value for that length). In some cases (lengths 13, 15, 19, and 20) the limiting Pursley-Sarwate criterion reported on Table 2 is lower. We were able to go to much higher lengths in Table 3 because the computational burden is greatly reduced when we restrict the calculations of crosscorrelation properties to only those pairs of seeds that have minimum autocorrelation. The lowest asymptotic value for the Pursley-Sarwate criterion we discovered was $331/300 = 1.10333\dots$ for $\ell = 40$ on Table 3. As far as the authors know, this is the lowest asymptotic value reported to date, and our construction can be compared to randomly selected long binary sequences, which typically have Pursley-Sarwate criterion of about 2, and to high-performance sequence pairs constructed from finite field characters that have asymptotic Pursley-Sarwate criterion of $7/6$ (see [9, §II.E, §IV.D] and [1, eq.(6)]).

Our computations of limiting crosscorrelation demerit factors in Tables 2 and 3 use a fast Fourier transform algorithm to speed up the convolutions (Laurent polynomial multiplications) that appear in the formula for asymptotic crosscorrelation demerit factor in Theorem 1.2. Because these calculations are performed using floating point arithmetic, there are small rounding errors. We checked that the approximate values of the norms and integrals in our formula were always very close to integers: all discrepancies were less than $3 \cdot 10^{-12}$.

TABLE 3. Lowest Limiting Pursley-Sarwate Criterion among Seed Pairs that Have the Lowest Limiting Autocorrelation Demerit Factor

seed length	limiting values as $n \rightarrow \infty$				orbit size	seeds	
	$\text{PSC}(f_n, g_n)$	$\text{ADF}(f_n)$	$\text{ADF}(g_n)$	$\text{CDF}(f_n, g_n)$		f_0	g_0
1	1.6666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	4	0	0
2	1.3333...	$\frac{1}{3}$	$\frac{1}{3}$	1	8	0	1
3	1.3703...	$\frac{17}{27}$	$\frac{17}{27}$	$\frac{20}{27}$	32	0	1
4	1.1666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	16	1	2
5	1.3466...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{4}{5}$	32	01	02
5	1.3466...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{4}{5}$	32	01	08
5	1.3466...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{4}{5}$	32	01	0D
6	1.2962...	$\frac{17}{27}$	$\frac{17}{27}$	$\frac{2}{3}$	32	02	0D
6	1.2962...	$\frac{17}{27}$	$\frac{17}{27}$	$\frac{2}{3}$	32	04	0B
7	1.2312...	$\frac{73}{147}$	$\frac{73}{147}$	$\frac{36}{49}$	32	04	1A
8	1.1666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	32	06	3A
8	1.1666...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	32	12	2E
9	1.2057...	$\frac{113}{243}$	$\frac{113}{243}$	$\frac{20}{27}$	32	009	035
10	1.1733...	$\frac{41}{75}$	$\frac{41}{75}$	$\frac{47}{75}$	32	04D	0A1
11	1.1818...	$\frac{161}{363}$	$\frac{161}{363}$	$\frac{268}{363}$	32	032	251
12	1.1666...	$\frac{11}{27}$	$\frac{11}{27}$	$\frac{41}{54}$	32	065	6A3
13	1.1775...	$\frac{217}{507}$	$\frac{217}{507}$	$\frac{380}{507}$	32	01DB	0D47
14	1.1836...	$\frac{73}{147}$	$\frac{73}{147}$	$\frac{101}{147}$	32	0071	149B
15	1.1688...	$\frac{281}{675}$	$\frac{281}{675}$	$\frac{508}{675}$	32	01AC	245C
16	1.1041...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{37}{48}$	32	0A36	11D2
17	1.1407...	$\frac{353}{867}$	$\frac{353}{867}$	$\frac{212}{289}$	32	0038D	0EE96
18	1.1481...	$\frac{113}{243}$	$\frac{113}{243}$	$\frac{166}{243}$	32	0039A	0E8F6
19	1.1643...	$\frac{433}{1083}$	$\frac{433}{1083}$	$\frac{276}{361}$	32	00F26	0C549
20	1.14	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{121}{150}$	16	05239	36E0A

TABLE 3. (continued) Lowest Limiting Pursley-Sarwate Criterion among Seed Pairs that Have the Lowest Limiting Autocorrelation Demerit Factor

seed length	limiting values as $n \rightarrow \infty$				orbit size	seeds	
	$\text{PSC}(f_n, g_n)$	$\text{ADF}(f_n)$	$\text{ADF}(g_n)$	$\text{CDF}(f_n, g_n)$		f_0	g_0
21	1.1405...	$\frac{521}{1323}$	$\frac{521}{1323}$	$\frac{988}{1323}$	32	001C9A	063EAD
22	1.1515...	$\frac{161}{363}$	$\frac{161}{363}$	$\frac{257}{363}$	32	0188B5	1341DE
22	1.1515...	$\frac{161}{363}$	$\frac{161}{363}$	$\frac{257}{363}$	32	022D85	0C74FD
23	1.1424...	$\frac{617}{1587}$	$\frac{617}{1587}$	$\frac{52}{69}$	16	00ECB6	1C312A
23	1.1424...	$\frac{617}{1587}$	$\frac{617}{1587}$	$\frac{52}{69}$	16	08643E	14B9A2
24	1.1296...	$\frac{19}{54}$	$\frac{19}{54}$	$\frac{7}{9}$	16	032695	03CE6A
25	1.1546...	$\frac{721}{1875}$	$\frac{721}{1875}$	$\frac{1444}{1875}$	32	003B3CB	04E50A2
25	1.1546...	$\frac{721}{1875}$	$\frac{721}{1875}$	$\frac{1444}{1875}$	32	01FAE32	0C42A69
26	1.1360...	$\frac{217}{507}$	$\frac{217}{507}$	$\frac{359}{507}$	32	042347C	0A6B813
27	1.1545...	$\frac{833}{2187}$	$\frac{833}{2187}$	$\frac{188}{243}$	32	0A109EC	3E2ACD6
28	1.1326...	$\frac{53}{147}$	$\frac{53}{147}$	$\frac{227}{294}$	32	071DBB5	3B82B6D
29	1.1625...	$\frac{953}{2523}$	$\frac{953}{2523}$	$\frac{660}{841}$	32	0089E14E	064BADE8
29	1.1625...	$\frac{953}{2523}$	$\frac{953}{2523}$	$\frac{660}{841}$	32	013A6A2D	021C14EC
30	1.1288...	$\frac{281}{675}$	$\frac{281}{675}$	$\frac{481}{675}$	32	02A6CF21	10164AC7
31	1.1394...	$\frac{1081}{2883}$	$\frac{1081}{2883}$	$\frac{2204}{2883}$	32	067E2CAB	13AF5F63
32	1.1041...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{37}{48}$	32	0A363905	11D21EDD
32	1.1041...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{37}{48}$	32	1EDD11D2	39050A36
33	1.1303...	$\frac{1217}{3267}$	$\frac{1217}{3267}$	$\frac{2476}{3267}$	32	00C03B652	0B5B9C517
34	1.1107...	$\frac{353}{867}$	$\frac{353}{867}$	$\frac{610}{867}$	32	00598B0ED	0DEE9382B
35	1.1746...	$\frac{1361}{3675}$	$\frac{1361}{3675}$	$\frac{2956}{3675}$	32	00963532E	1E280FB3B
36	1.1872...	$\frac{29}{81}$	$\frac{29}{81}$	$\frac{403}{486}$	16	1D603A324	7190953ED
36	1.1872...	$\frac{29}{81}$	$\frac{29}{81}$	$\frac{403}{486}$	16	1DA30602B	7EACA6F12
37	1.1796...	$\frac{1513}{4107}$	$\frac{1513}{4107}$	$\frac{3332}{4107}$	16	00035AC726	0636C1F2AA
37	1.1796...	$\frac{1513}{4107}$	$\frac{1513}{4107}$	$\frac{3332}{4107}$	16	0223D0E7AE	04164BD222
38	1.1209...	$\frac{433}{1083}$	$\frac{433}{1083}$	$\frac{781}{1083}$	32	0210F456C9	1A2E842E73
39	1.1477...	$\frac{1673}{4563}$	$\frac{1673}{4563}$	$\frac{132}{169}$	32	0019E3352D	07819B4CAA
39	1.1477...	$\frac{1673}{4563}$	$\frac{1673}{4563}$	$\frac{132}{169}$	32	166AAF0FCC	195A5FFCC3
40	1.1033...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{77}{100}$	16	0033C66A5A	0F03369955
40	1.1033...	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{77}{100}$	16	33F0F55669	3CC005A566

REFERENCES

- [1] T. Boothby and D. J. Katz. Low correlation sequences from linear combinations of characters. *preprint*, arXiv:1602.04514 [cs.IT], 2016.
- [2] P. Borwein and M. Mossinghoff. Rudin-Shapiro-like polynomials in L_4 . *Math. Comp.*, 69(231):1157–1166, 2000.
- [3] M. Golay. A class of finite binary sequences with alternate auto-correlation values equal to zero. *IEEE Trans. Inform. Theory*, 18(3):449–450, 1972.
- [4] S. W. Golomb and G. Gong. *Signal design for good correlation*. Cambridge University Press, Cambridge, 2005.
- [5] T. Høholdt. The merit factor problem for binary sequences. In *Applied algebra, algebraic algorithms and error-correcting codes*, volume 3857 of *Lecture Notes in Comput. Sci.*, pages 51–59. Springer, Berlin, 2006.
- [6] T. Høholdt and H. E. Jensen. Determination of the merit factor of Legendre sequences. *IEEE Trans. Inform. Theory*, 34(1):161–164, 1988.
- [7] J. Jedwab. A survey of the merit factor problem for binary sequences. In T. Helleseth, D. Sarwate, H.-Y. Song, and K. Yang, editors, *Sequences and Their Applications - SETA 2004*, volume 3486 of *Lecture Notes in Computer Science*, pages 19–21. Springer Berlin / Heidelberg, 2005.
- [8] K. H. A. Kärkkäinen. Mean-square cross-correlation as a performance measure for department of spreading code families. In *IEEE Second International Symposium on Spread Spectrum Techniques and Applications*, pages 147–150, 1992.
- [9] D. J. Katz. Aperiodic crosscorrelation of sequences derived from characters. *IEEE Transactions on Information Theory*, 62(9):5237–5259, 2016.
- [10] J. E. Littlewood. On polynomials $\sum^n \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta i}$. *J. London Math. Soc.*, 41:367–376, 1966.
- [11] J. E. Littlewood. *Some Problems in Real and Complex Analysis*. D. C. Heath and Co. Raytheon Education Co., Lexington, MA, 1968.
- [12] M. B. Pursley and D. V. Sarwate. Bounds on aperiodic cross-correlation for binary sequences. *Electronics Letters*, 12(12):304–305, 1976.
- [13] W. Rudin. Some theorems on Fourier coefficients. *Proc. Amer. Math. Soc.*, 10:855–859, 1959.
- [14] D. V. Sarwate and M. B. Pursley. Crosscorrelation properties of pseudorandom and related sequences. *IEEE Trans. Inform. Theory*, 68(5):593–619, 1980. Correction in *IEEE Trans. Inform. Theory* 68(12):1554, 1980.
- [15] R. A. Scholtz and L. R. Welch. Group characters: sequences with good correlation properties. *IEEE Trans. Inform. Theory*, 24(5):537–545, 1978.
- [16] M. R. Schroeder. *Number Theory in Science and Communication*, volume 7 of *Springer Series in Information Sciences*. Springer-Verlag, Berlin, fourth edition, 2006.
- [17] H. S. Shapiro. Extremal problems for polynomials and power series. Master’s thesis, Dept. of Mathematics, Massachusetts Institute of Technology, Cambridge, 1951.

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